Anomalous scalings and dynamics of magnetic helicity

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It is demonstrated that the two-point correlation function of the magnetic helicity in the case of zero mean magnetic field has anomalous scalings for both compressible and incompressible turbulent helical fluid flow. The magnetic helicity in the limit of very high electrical conductivity is conserved. This implies that the two-point correlation function of the conserved property does not necessarily have normal scaling. The reason for the anomalous scalings of the magnetic helicity correlation function is that the magnetic field in the equation for the two-point correlation function of the magnetic helicity plays a role of a pumping with anomalous scalings. It is shown also that when magnetic fluctuations with zero mean magnetic field are generated the magnetic helicity is very small even if the hydrodynamic helicity is large.
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I. INTRODUCTION

Problems of intermittency and anomalous scalings for vector (magnetic) and scalar fields passively advected by a turbulent fluid flow are a subject of active research in the last years (see, e.g., [1–8]). The anomalous scaling means the deviation of the scaling exponents of the correlation function of a vector (scalar) field from their values obtained by the dimensional analysis. An interesting question is the role of the conservation laws in the problem of intermittency and anomalous scalings. For the passive scalar advected by incompressible and homogeneous turbulent fluid flow the quantity n^2 (or T^2) is conserved (for infinitely small diffusivity or thermal conductivity), where n is the number density of particles, and T is the fluid temperature. Corresponding two-point correlation function $\langle n(t,\mathbf{x}) n(t,\mathbf{y})\rangle$ has normal scaling (see, e.g., [2,3]). For the passive vector (magnetic field) the quantity h^2 is not conserved and the second moment $\langle \mathbf{h}(t,\mathbf{x})\cdot\mathbf{h}(t,\mathbf{y})\rangle$ has anomalous scalings [4,7], where $\mathbf{h}$ is the magnetic field. In this case the total (magnetic plus hydrodynamic) energy is conserved. On the other hand, the magnetic helicity $\langle \mathbf{a}(t,\mathbf{x})\cdot\mathbf{h}(t,\mathbf{x})\rangle$ in the limit of very high electrical conductivity is conserved. Here $\mathbf{a}$ is a vector potential of magnetic field, i.e., $\mathbf{h}=\boldsymbol{\nabla}\times\mathbf{a}$. What is a scaling for the two-point correlation function $\langle \mathbf{a}(t,\mathbf{x})\cdot\mathbf{h}(t,\mathbf{y})\rangle$ of the magnetic helicity?

In this paper we show that the two-point correlation function of magnetic helicity has anomalous scalings for both, compressible and incompressible turbulent helical fluid flow. For the helical fluid velocity field $\alpha(r)\neq 0$ [see Eq. (3) below]. The reason for the anomalous scalings is that the magnetic field in the equation for the two-point correlation function of magnetic helicity plays a role of a pumping with anomalous scalings. This implies that the two-point correlation function of the conserved property does not necessarily have normal scaling. This demonstrates a difference between passive scalar and vector fields. We also study here an excitation of the magnetic helicity by a helical turbulent fluid flow in the case of generation of magnetic fluctuations with zero mean magnetic field. Note that for a nonhelical velocity field [$\alpha(r)=0$, see Eq. (3) below] the scaling exponent of the magnetic helicity correlation function is normal, i.e., it can be obtained from the dimensional arguments [5].

II. GOVERNING EQUATIONS

We study the evolution of magnetic fluctuations with zero mean magnetic field in a low-Mach-number compressible turbulent fluid flow. A mechanism of the generation of magnetic fluctuations with a zero mean magnetic field was proposed in [9] and comprises stretching, twisting and folding of the original loop of a magnetic field. These nontrivial motions are three dimensional and result in an amplification of the magnetic field. The magnetic field is determined by the induction equation $\partial\mathbf{h}/\partial t+(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{h}=(\mathbf{h}\cdot\boldsymbol{\nabla})\mathbf{u}-\mathbf{h}(\boldsymbol{\nabla}\cdot\mathbf{u})+\eta\Delta\mathbf{h}$, where $\mathbf{u}$ is the fluid velocity, η is the magnetic diffusion. We derive equations for the second-order correlation functions of the magnetic field and the magnetic helicity and we use a method of path integrals and modified Feynman-Kac formula (see, e.g., [1,6–8,11]). The equation for the second-order correlation function $h_{ij}=\langle h_i(t,\mathbf{x})h_j(t,\mathbf{y})\rangle$ of the magnetic field is

$$\partial h_{ij}/\partial t=[\hat{L}_{ik}(\mathbf{x})\delta_{js}+\hat{L}_{js}(\mathbf{y})\delta_{ik}+\hat{N}_{ijks}]h_{ks}+I_{ij} \tag{1}$$

(for details see [7]), where

$$\hat{L}_{ij}=\varepsilon_{iks}\frac{\partial}{\partial x_k}\left[\varepsilon_{smj}U_m+\alpha_{sj}-\hat{\eta}_{sm}\varepsilon_{mpj}\frac{\partial}{\partial x_p}\right],$$

$$\frac{1}{2}\hat{N}_{ijks}=\delta_{ik}\delta_{js}f_{mn}\frac{\partial^2}{\partial x_m\partial y_n}+\frac{\partial^2 f_{ij}}{\partial x_k\partial y_s}-\delta_{ik}\frac{\partial f_{mj}}{\partial y_s}\frac{\partial}{\partial x_m}$$
$$-\delta_{js}\frac{\partial f_{in}}{\partial x_k}\frac{\partial}{\partial y_n}+\delta_{ik}\delta_{js}\frac{\partial f_{mp}}{\partial y_p}\frac{\partial}{\partial x_m}+\delta_{ik}\delta_{js}\frac{\partial f_{pn}}{\partial x_p}\frac{\partial}{\partial y_n}$$
$$-\delta_{ik}\frac{\partial^2 f_{pj}}{\partial x_p\partial y_s}-\delta_{js}\frac{\partial^2 f_{ip}}{\partial x_k\partial y_p}+\delta_{ik}\delta_{js}\frac{\partial^2 f_{pl}}{\partial x_p\partial y_l},$$

and $f_{mn}=\langle \tau u_m(\mathbf{x})u_n(\mathbf{y})\rangle$, and δ_{mn} is the Kronecker tensor and ε_{ikm} is the Levi-Civita tensor, $\hat{\eta}_{ij}=(\eta_{pp}\delta_{ij}-\eta_{ij})/2$, and $\mathbf{U}=\mathbf{V}-\boldsymbol{\nabla}_p\langle\tau u_p\mathbf{u}\rangle/2$, and $\alpha_{mn}=-(\varepsilon_{mji}\langle\tau u_i\boldsymbol{\nabla}_n u_j\rangle$

$+\varepsilon_{nji}\langle\tau u_i \boldsymbol{\nabla}_m u_j\rangle)/2$, and $\eta_{pm}=\eta\delta_{pm}+\langle\tau u_p u_m\rangle$, and $\tau(r)$ is the scale-dependent momentum relaxation time, $\mathbf{V}=\langle\mathbf{u}\rangle$, the tensor I_{ij} is determined by an external source of magnetic fluctuations, $\mathbf{r}=\mathbf{y}-\mathbf{x}$. We seek a solution for the second moment of the magnetic field in the form

$$\langle h_m(\mathbf{x})h_n(\mathbf{x}+\mathbf{r})\rangle = W(r)\delta_{mn}+(rW'/2)P_{mn} + \mu(r)\varepsilon_{mnp}r_p/2, \quad (2)$$

where $P_{mn}=\delta_{mn}-r_m r_n/r^2$, and $W'=dW/dr$. Note that the current helicity correlation function $\mu(r)=[\langle\mathbf{h}(\mathbf{x})\cdot(\vec{\boldsymbol{\nabla}}\times\mathbf{h}(\mathbf{y}))\rangle+\langle\mathbf{h}(\mathbf{y})\cdot(\vec{\boldsymbol{\nabla}}\times\mathbf{h}(\mathbf{x}))\rangle]/2=\hat{S}\chi/3$, where the magnetic helicity correlation function $\chi(r)=[\langle\mathbf{a}(\mathbf{x})\cdot\mathbf{h}(\mathbf{y})\rangle+\langle\mathbf{h}(\mathbf{x})\cdot\mathbf{a}(\mathbf{y})\rangle]/2$, and $\hat{S}\chi=\chi''+4\chi'/r$. The correlation function $\langle\tau u_m u_n\rangle$ is

$$\langle\tau u_m(\mathbf{x})u_n(\mathbf{x}+\mathbf{r})\rangle=\eta_T[(F+F_c)\delta_{mn}+rF'P_{mn}/2 + F_c' r_m r_n/r+\alpha(r)\varepsilon_{mnp}r_p/2] \quad (3)$$

(see [8]), where $\eta_T=u_0^2\tau_0/3$ is the turbulent magnetic diffusion, u_0 is the characteristic velocity in the maximum scale l_0 of turbulent motions, $\tau_0=l_0/u_0$, $F(0)=1-F_c(0)$. The function $F_c(r)$ describes the compressible (potential) component, whereas $F(r)$ corresponds to the vortical part of the turbulence. The function $\alpha(r)=-[\langle\tau\mathbf{u}(\mathbf{x})\cdot(\vec{\boldsymbol{\nabla}}\times\mathbf{u}(\mathbf{y}))\rangle+\langle\tau\mathbf{u}(\mathbf{y})\cdot(\vec{\boldsymbol{\nabla}}\times\mathbf{u}(\mathbf{x}))\rangle](6u_0)^{-1}$. In Eqs. (2) and (3) the dimensionless distance r is measured in the units l_0.

We use here the δ-correlated in time random process to describe a turbulent velocity field. Using the δ-correlated-in-time random process allows us to obtain the analytical results for the anomalous scalings of the two-point correlation functions of the magnetic field and magnetic helicity. The results remain valid also for the velocity field with a finite correlation time if the second-order correlation functions of the magnetic field and magnetic helicity vary slowly in comparison to the correlation time of the turbulent velocity field (see, e.g., [1]). We also take into account the dependence of the momentum relaxation time on the scale of the turbulent velocity field: $\tau(\mathbf{k})=\tau_0 k^{1-p}$, where p is the exponent in spectrum of kinetic turbulent energy, k is the wave number measured in the units l_0^{-1}.

Using Eqs. (1)–(3) we derive equations for the correlation functions of the magnetic field $W(t,r)$ and the magnetic helicity $\chi(t,r)$. Indeed,

$$\partial W/\partial t=(W''+\zeta W'-\xi W)/m-2(\alpha_0-\alpha(r))\mu+\tilde{I}, \quad (4)$$

$$\partial\mu/\partial t=\hat{S}[2(\alpha_0-\alpha(r))W+\mu/m], \quad (5)$$

where $\alpha_0=\alpha(r=0)=-\langle\tau\mathbf{u}\cdot(\vec{\boldsymbol{\nabla}}\times\mathbf{u})\rangle/3$ is the α-effect, $\tilde{I}$ is an external source of magnetic fluctuations, and $1/m=2/\mathrm{Rm}+2[1-F-(rF_c)']/3$, and $\zeta=4/r+m(1/m)'$, and $\xi=2m(f'+2f_c')/r$, and $f=F+rF'/3$, and $f_c=F_c+rF_c'/3$, and $\mathrm{Rm}=u_0 l_0/\eta\gg 1$ is the magnetic Reynolds number, and the functions $F(r)$ and $F_c(r)$ are determined below. Equations (4) and (5) are written in dimensionless variables: coordinates and time are measured in the units l_0 and τ_0, the velocity is measured in the units u_0, the magnetic field is measured in the units B_0. Note that in [12] the system of equations which is similar to Eqs. (4) and (5) was derived. However, there are mistakes in the equations derived in [12]. Since $\mu(r)=\hat{S}\chi/3$, Eq. (5) can be rewritten as

$$\partial\chi/\partial t=(\chi''+4\chi'/r)/m+6(\alpha_0-\alpha(r))W. \quad (6)$$

Equation (6) at $r=0$ is given by $(\partial\chi/\partial t)_{r=0}=2(\chi''+4\chi'/r)_{r=0}/\mathrm{Rm}$. This implies that in a very high electrical conductivity limit ($\mathrm{Rm}\to\infty$) the magnetic helicity $\chi(r=0)\equiv\langle\mathbf{a}(\mathbf{x})\cdot\mathbf{h}(\mathbf{x})\rangle$ is conserved. We seek a solution of Eqs. (4) and (6) in the form: $W(t,r)=(\Psi(r)\sqrt{m}/r^2)\exp(\Gamma t)$ and $\chi(t,r)=(\kappa(r)/r^2)\exp(\Gamma t)$, where the functions $\Psi(r)$ and $\kappa(r)$ are determined by

$$\Psi''/m(r)-[\Gamma+U(r)]\Psi=v(r)[\kappa''-2\kappa/r^2]/9m+I, \quad (7)$$

$$\kappa''/m(r)-[\Gamma+2/(mr^2)]\kappa=-v(r)\Psi, \quad (8)$$

and $v(r)=6\sqrt{m}(\alpha_0-\alpha(r))$, and $I=r^2\tilde{I}/\sqrt{m}$, and $U(r)=(\zeta^2+2\zeta'+4\xi)/4m(r)$. We consider the case of small magnetic Prandtl numbers $\mathrm{Pr}_m=\nu/\eta\ll 1$, which is typical for many astrophysical and geophysical applications, where ν is the kinematic viscosity. We choose the following model of turbulence. Incompressible $F(r)$ and compressible $F_c(r)$ components in the inertial range of turbulence $r_d<r<1$ are given by $F(r)=(1-r^{q-1})/(1+\sigma)$, $F_c(r)=(1-r^{q-1})\sigma/(1+\sigma)$, where σ is the degree of compressibility, q is the exponent in spectrum of the function $\langle\tau u_m u_n\rangle$, and $r_d=\mathrm{Re}^{-1/(3-p)}$, p is the exponent in the spectrum of kinetic turbulent energy, and $\mathrm{Re}=u_0 l_0/\nu\gg 1$ is the Reynolds number. Note that the exponent p in the spectrum of kinetic turbulent energy differs from that of the function $\langle\tau u_m u_n\rangle$ due to the scale dependence of the momentum relaxation time τ of the turbulent velocity field. The relation between p and q is $q=2p-1$ [7]. Equation (4) for $\alpha(r)=0$ and $\sigma=0$ was derived in [10].

The solution of Eqs. (7) and (8) can be obtained using an asymptotic analysis (see, e.g., [1,6–8]). This analysis is based on the separation of scales. In particular, the solutions of the Schrödinger equations (7) and (8) with a variable mass have different regions where the form of the potential $U(r)$, mass $m(r)$ and, therefore, eigenfunctions $\Psi(r)$ and $\kappa(r)$ are different. Solutions in these different regions can be matched at their boundaries. The results obtained by this asymptotic analysis are presented below.

III. ANOMALOUS SCALINGS

We study a zero mode, i.e., we obtain the solutions of Eqs. (7) and (8) at $\Gamma=0$. In this case Eqs. (7) and (8) are given by

$$\Psi''/m(r)-\tilde{U}(r)\Psi=I, \quad (9)$$

$$\kappa''-2\kappa/r^2=-v(r)m\Psi, \quad (10)$$

where $\tilde{U}(r)=U(r)-4m(\alpha_0-\alpha(r))^2$, and the function $\alpha(r)=\alpha_0(1-r^{q-1})$ for $0\leqslant r\leqslant 1$, and $\alpha(r)=0$ for $r>1$, and the external source of magnetic fluctuations $\tilde{I}(r)=I_0(1-r^s)$ for $0\leqslant r\leqslant 1$, and $\tilde{I}(r)=0$ for $r>1$, and $s>0$. The solutions of Eqs. (9) and (10) have three characteristic regions. In region I, i.e., for $0 \leqslant r \leqslant \mathrm{Rm}^{-1/(q-1)}$, the functions $W(r)$ and $\chi(r)$ are given by $W(r) = I_*(1-\beta_0 \, \mathrm{Rm} \, r^{q-1})$, and $\chi(r) = B_1 + a_1 \alpha_0 \, \mathrm{Rm} \, r^{q+1}$, where $\beta_0 = \beta_m + \xi_0(q-1)/(q+2)$, and $I_* \sim I_0 \, \mathrm{Rm}^{(q+2)/2(q-1)}$, and $a_1 = A_1/[(q+1)(q+4)]$, and $\beta_m = (1+q\sigma)/3(1+\sigma)$, and $\xi_0 = (1+2\sigma)(2+q)(q-1)/3(1+\sigma)$. In region II, i.e., for $\mathrm{Rm}^{-1/(q-1)} \ll r \ll 1$, the functions $W(r)$ and $\chi(r)$ are given by

$$W(r) = A_2 m^{1/2} r^{-3/2} \cos(b \ln r + \varphi_0) + W_N, \tag{11}$$

$$\chi(r) = B_2 + B_3/r^3 + a_2 \alpha_0 \, \mathrm{Re}\{r^{-(q/2-1+ib)}\}, \tag{12}$$

where $W_N = -I_0 r^{3-q}/[2\beta_m((4-q/2)^2 + |b|^2)]$, and

$$b^2 = \left(\frac{q^2-4}{4}\right)\left(\frac{3+\sigma(4-q)}{1+q\sigma}\right),$$

$$a_2 = \frac{(3A_2/\sqrt{2})\beta_m^{-3/2}}{[((1-q/2)^2+|b|^2)((4-q/2)^2+|b|^2)]^{1/2}}.$$

Note that when $q \geqslant 2$ the parameter b is a complex number and $\mathrm{Re}\{r^{-(q/2-1+ib)}\} = r^{-q/2+1}\cos(b \ln r)$, and when $q<2$ the parameter b is a real number. For $q<2$ the solution for $W(r)$ is given by $W(r) = m^{1/2} r^{-3/2}(A_2 r^{-|b|} + A_4 r^{|b|})$. In region III ($r \gg 1$), the functions $W(r) = A_3 r^{-2}(3\alpha_0 \cos(3\alpha_0 r) - r^{-1}\sin(3\alpha_0 r))$, and $\chi(r) = B_4/r^3 - 6A_3\alpha_0 r^{-1}\sin(3\alpha_0 r)$, where we take into account the boundary condition for the function $\chi(r)$, i.e., $\chi(r) \to 0$ for $r \to \infty$, and a condition $\chi(r) \to 0$ for $\alpha_0 \to 0$. Matching the functions $W(r)$ and $W'(r)$ at the boundaries of these regions yields $A_1 \sim A_2 \sim A_3 \sim A_4 \sim I_0$. On the other hand, matching the functions $\chi(r)$ and $\chi'(r)$ at the boundaries of the regions yields $B_1 \sim I_0 \alpha_0 \, \mathrm{Rm}^{(q-2)/2(q-1)}$ for $2 \leqslant q \leqslant 3$, and $B_1 \sim I_0 \alpha_0 \, \mathrm{Rm}^{(q-2+2b)/2(q-1)}$ for $1<q<2$, and $B_2 \sim I_0 \alpha_0$ for $1<q \leqslant 3$, and $B_3 \sim I_0 \alpha_0 \, \mathrm{Rm}^{-(8-q)/2(q-1)}$ for $2 \leqslant q \leqslant 3$, and $B_3 \sim I_0 \alpha_0 \, \mathrm{Rm}^{(q-6+2b)/2(q-1)}$ for $1<q<2$.

The magnetic fluctuations are excited when the magnetic Reynolds number $\mathrm{Rm} > \mathrm{Rm}^{\mathrm{cr}}$, where the critical magnetic Reynolds number $\mathrm{Rm}^{\mathrm{cr}}$ is found in [7]. For incompressible fluid $\sigma=0$ and $p=5/3$ (Kolmogorov turbulence) the critical magnetic Reynolds number $\mathrm{Rm}^{(\mathrm{cr})} = 412$, while for compressible fluid flow $\sigma=0.1$ the value $\mathrm{Rm}^{(\mathrm{cr})} = 740$. For a larger parameter of compressibility the critical magnetic Reynolds number increases sharply up to $\mathrm{Rm}^{(\mathrm{cr})} \sim 10^6$ (see, [7]). First, we consider the case $\mathrm{Rm} < \mathrm{Rm}^{\mathrm{cr}}$, i.e., when there is no self-excitation of the magnetic fluctuations. In this case the magnetic fluctuations are sustained by an external source. The first term in Eq. (11) for the correlation function of the magnetic field $W(r)$ in the inertial range is given by $W_A \sim r^{-q/2-1}\cos(b \ln r + \varphi_0)$. This corresponds to the anomalous scaling of the magnetic fluctuations. The normal scaling for the second moment of the magnetic fluctuations is given by the second term in Eq. (11): $W_N \sim r^{3-q}$. The general solution of equation for the second-order correlation function of magnetic field $W(r)$ includes solutions describing the anomalous and normal scalings. The anomalous scaling $W_A \sim r^{-q/2-1}\cos(b \ln r + \varphi_0)$ can be presented as the real part of the power-law function r^{ϵ} with the complex exponent $\epsilon = -q/2 - 1 + ib(\sigma,q)$. This anomalous scaling corresponds to the deviation from the condition of the constant flux of magnetic fluctuations over the spectrum. It describes the case $2<q<3$. When $1<q<2$ the anomalous exponent in a low-Mach-number compressible turbulent flow is real, i.e., $\epsilon = -q/2 - 1 + |b(\sigma,q)|$. In the case of incompressible turbulent flow ($\sigma=0$) and $1<q<2$ this result coincides with that obtained in [4]. For incompressible turbulent flow and $2<q<3$ the anomalous scaling of magnetic field is the complex number $\epsilon = -q/2 - 1 - ib$ ($\sigma=0,q$), see [7].

Now we discuss solutions for the correlation function of the magnetic helicity. The last term in Eq. (12) $\propto \mathrm{Re}\{r^{-(q/2-1+ib)}\}$ corresponds to the anomalous scaling for the magnetic helicity correlation function. The magnetic helicity $\chi_0 = \chi$ ($r=0$) in the limit of very large electrical conductivity is conserved. This means that the two-point correlation function of conserved property has anomalous scaling. The reason for the anomalous scalings of the magnetic helicity correlation function is that the magnetic field in the equation for the two-point correlation function of the magnetic helicity plays a role of a pumping with anomalous scalings [see Eqs. (9) and (10)]. This demonstrates a difference between passive vector (magnetic field) and passive scalar. Indeed, the quantity n^2 (or T^2) in incompressible and homogeneous turbulent fluid flow is conserved (for infinitely small diffusivity or thermal conductivity). Corresponding two-point correlation function has normal scaling. On the other hand, the magnetic helicity is conserved and the two-point correlation function of magnetic helicity has anomalous scaling for a turbulent helical velocity field. Thus we demonstrated here that two-point correlation function of the conserved property does not necessarily have normal scaling. Note that the nonlinear effects (i.e., self-consistent dynamics in which the back-reaction of the Lorentz force is considered) are important when the amplitude of the magnetic field is enough large, i.e., when $\langle \mathbf{h}^2 \rangle / 4\pi \sim \langle \rho \mathbf{u}^2 \rangle$. In this section we consider the case when there is no magnetic dynamo, i.e., the magnetic fluctuations are sustained by an external source I. The obtained results are valid when the external source I is not very strong, i.e., $I\tau/4\pi \ll \langle \rho \mathbf{u}^2 \rangle$, and the nonlinear effects are not important.

IV. DYNAMICS OF MAGNETIC HELICITY

Now, we consider the case $\mathrm{Rm} > \mathrm{Rm}^{\mathrm{cr}}$, i.e., when the magnetic fluctuations with zero mean magnetic field are excited. What is the dynamics of the magnetic helicity in a helical turbulent fluid flow? Equations (7) and (8) can be rewritten in the form $(\hat{Q} + \hat{V})\mathbf{X} = \Gamma \mathbf{X}$, where $\mathbf{X}$ is the vector-colomn with the components $X_1 = \Psi$, and $X_2 = \kappa$, and the matrix $\hat{Q} \equiv Q_{ij} = 0$, when $i \neq j$, and $Q_{11} = (d^2/dr^2 - U)/m$, and $Q_{22} = (d^2/dr^2 - 2/r^2)/m$, and the matrix $\hat{V} \equiv V_{ij} = 0$, when $i=j$, and $V_{12} = -(d^2/dr^2 - 2/r^2)/9m$, and $V_{21} = 1$. We consider the modes which satisfy the following property: the change $\alpha(r) \to -\alpha(r)$ in Eqs. (7) and (8) results in the change $\kappa(r) \to -\kappa(r)$ and $\Psi(r) \to \Psi(r)$. We seek a solution of this equation in the form $\mathbf{X} = \sum_{k=1}^{\infty} x_k \mathbf{e}_k + \int y(\gamma) \mathbf{E}(\gamma) d\gamma$, where the eigenfunctions $\mathbf{e}_k$ and $\mathbf{E}(\gamma)$ satisfy to the equations: $\hat{Q}\mathbf{e}_k = \gamma_k \mathbf{e}_k$, and $\hat{Q}\mathbf{E}(\gamma) = \gamma \mathbf{E}(\gamma)$. Here $\mathbf{e}_k$ is the vector-colomn with the components $e_{1k} = \tilde{\Psi}_k$, and $e_{2k} = 0$, and $\mathbf{E}(\gamma)$ is the vector-colomn with the components $E_1=0$, and $E_2=\tilde{\kappa}(\gamma)$. The functions $\tilde{\Psi}_k(r)$ and $\tilde{\kappa}(r)$ are determined by Eqs. (7) and (8) with the condition $\alpha(r)=0$. The equation for the function $\mathbf{e}_k$ describes nonhelical component of the magnetic field correlation function and it has discrete spectrum. On the other hand, the equation for the function $\mathbf{E}(\gamma)$ determines helical component of the magnetic field correlation function and it has continuous spectrum. The continuous spectrum corresponds to $\gamma<0$, i.e., it describes the relaxation of the magnetic helicity correlation function. The discrete spectrum of the equation for the function $\mathbf{e}_k$ corresponds to the generation of the magnetic fluctuations ($\gamma_p>0$). The normalize conditions for the eigenfunctions $\mathbf{e}_k$ and $\mathbf{E}(\gamma)$ are given by $\int m(r)\mathbf{e}_k^\dagger \mathbf{e}_p dr =T(k)\delta_{kp}$, and $\int m(r)\mathbf{E}^\dagger(\gamma)\mathbf{E}(\gamma')dr=S(\gamma)\delta(\gamma-\gamma')$. The standard procedure used in quantum mechanics (see, e.g., [13]) yields the equations for the functions x_p and $y(\gamma)$, i.e., $(\gamma_p-\Gamma)x_p=L(\mathbf{y})$, and $(\Gamma-\gamma)y(\gamma)=N(\mathbf{x})$, where $N(\mathbf{x})=(1/S(\gamma))\int v(r)m(r)\tilde{\kappa}(r)\tilde{x}(r)dr$, and $L(\mathbf{y})=(\gamma/T(p))\int v(r)\tilde{\Psi}_k(r)m(r)\tilde{y}(r)dr$, and $\tilde{x}(r)=\Sigma_{k=1}^{\infty}x_k\tilde{\Psi}_k(r)$, and $\tilde{y}(r)=\int y(\gamma')\tilde{\kappa}(\gamma')d\gamma'$. Now we use a perturbation theory (see, e.g., [13]), i.e., we seek the solutions of the above equations in the form of series $Z=Z^{(0)}+\varepsilon Z^{(1)}+\varepsilon^2 Z^{(2)}+\cdots$, where $Z=x_p;y;\Gamma$. The small parameter is $\varepsilon\sim \mathrm{Rm}^{-(q+2)/2(q-1)}$ (see below). The perturbation theory yields $\Gamma^{(2k+1)}=0$, and $\Gamma^{(0)}=\gamma_p$, and $\Gamma^{(2)}=-L(\mathbf{y}^{(1)})/x_p^{(0)}$, and $y^{(2k)}=0$, and $y^{(1)}=N(\mathbf{x}^{(0)})/(\gamma_p-\gamma)$, and $x_p^{(2k+1)}=0$, and $x_p^{(0)}=1$, and $x_p^{(2)}$ is determined by equation $N(\mathbf{x}^{(2)})(\gamma_p-\gamma)=\Gamma^{(2)}y^{(1)}$, etc.

For nonhelical turbulence $[\alpha(r)=0]$ the helical $\mu(r)$ and nonhelical $W(r)$ parts of the magnetic field correlation function are decoupled. This implies that the magnetic helicity can only relaxate from the initial value. In helical turbulence $[\alpha(r)\neq 0]$ the magnetic helicity $\chi(r)$ depends on $W(r)$. This implies that the eigenfunction $\mathbf{e}_p$ of the discrete spectrum is modified, i.e., it has spiral and nonspiral components: $e_{1p}=\Psi_p$ and $e_{2p}=\kappa_p$, where $\kappa_p(r)=\int_0^\infty\tilde{\kappa}(-\lambda,r)V_p(\lambda)/[\gamma_p+\lambda]d\lambda+O(\varepsilon^3)$, and $\lambda=|\gamma|$, and $\Psi_p=\tilde{\Psi}_p+O(\varepsilon^2)$, and $V_p(\lambda)=(1/S(-\lambda))\int v(r')m(r')\tilde{\kappa}(-\lambda,r)\Psi_p(r')dr'$. Using this equation we calculate the magnetic helicity $\chi(r=0)$. The result is given by $\chi(r=0)\sim -4\alpha_0\tau_0\,\mathrm{Rm}^{-(q+2)/2(q-1)}W(r=0)$, where $W(r=0)=W_0\exp(\gamma_p t)$. For nonhelical turbulence there is only the relaxation of the initial magnetic helicity $\chi(t=0)\neq 0$. On the other hand, for helical turbulence the magnetic helicity is excited due to the growth of magnetic fluctuations. However, the magnitude of the magnetic helicity is very small even if the hydrodynamic helicity is large, i.e., $\chi(r=0)=-4\alpha_0\tau_0W(r=0)/\mathrm{Rm}^{13/10}$, where we use the Kolmogorov spectrum $p=5/3$. The realizability condition $M(k)\geqslant|\chi(k)|k$ (see, e.g., [14]) allows us to estimate the maximum possible value of the magnetic helicity: $\chi_{\max}\sim l_0W\,\mathrm{Rm}^{-1/(q-1)}$, where $\langle\mathbf{h}^2\rangle=\int M(k)dk$. Therefore, $\chi(r=0)/\chi_{\max}\sim(\alpha_0/u_0)\mathrm{Rm}^{-q/2(q-1)}\ll 1$. This means that the magnetic helicity with zero mean magnetic field is very small even if the hydrodynamic helicity is large. Only nonzero mean magnetic field can create large magnetic helicity (see, e.g., [15]).

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